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NONLINEAR IMPLICIT FRACTIONAL DIFFERENTIAL EQUATION INVOLVING $\psi-$CAPUTO FRACTIONAL DERIVATIVE

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Abstract. In this paper, we consider a nonlinear implicit fractional differential equation with nonlocal condition and involving the Caputo fractional derivative with respect to another function. We investigate the existence, uniqueness of solution on subinterval of the original interval. Hence we give an estimation for this solution. Further, we discuss the continuous dependence of solution involved in the problem. The results obtained by means of a variety of tools fractional calculus including Banach contraction mapping principle. Illustrative examples are also given.

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1. Introduction

Kilbas et al. in [7] introduced the properties of fractional integrals and fractional derivatives of a function with respect to another function. Some of generalized fractional integral and differential operators and their properties were introduced by Agrawal in [1]. Recently, Almeida in [2] presented a new type of fractional differentiation operator called ψ- Caputo fractional operator and extended work of the Caputo [7, 8]. Almeida et al. in [3] investigated the existence and uniqueness results of nonlinear fractional differential equations involving a Caputo-type fractional derivative with respect to another function by means of fixed point theorem and Picard iteration method. Very recently, Dong et al. in [4] obtained some existence, and uniqueness of solution to an implicit fractional differential equation involving the Caputo fractional derivative

$$^{c}D_{0^+}^{\alpha}u(t) = f(t, u(t), {}^{c}D_{0^+}^{\alpha}u(t)), \quad t \in [0, b], \tag{1}$$

$$u(0) = u_0, \tag{2}$$

Haoues et al. in [6] established the existence and uniqueness of solution on a subinterval of the interval $[0, b]$ for a nonlinear implicit Caputo fractional differential equation (1) with the nonlocal condition $u(0) + g(u) = u_0$.

In this paper, we investigate the existence, uniqueness and continuous dependence of solutions on a subinterval of $[0, b]$ for the following nonlinear

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implicit fractional differential equation involving generalized Caputo fractional derivatives with respect to function ψ

$$(3) \qquad {}^{c}D_{0^+}^{\alpha,\psi}u(t) = f(t, u(t), {}^{c}D_{0^+}^{\alpha,\psi}u(t)), \quad t \in [0, b],$$

supplemented with the nonlocal condition

$$(4) \qquad u(0) + g(u) = u_0,$$

where $0 < \alpha < 1$, $u_0 \in \mathbb{R}$, ${}^{c}D_{0^+}^{\alpha,\psi}$ denotes the $\psi-$fractional derivative of order α in the sense of Caputo, $f : [0, b] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$, $g : C([0, b], \mathbb{R}) \to \mathbb{R}$ are nonlinear continuous functions satisfies some assumptions that will be specifed in Section 3, and $u \in C([0, b], \mathbb{R})$ such that the operator ${}^{c}D_{0^+}^{\alpha,\psi}$ exists and ${}^{c}D_{0^+}^{\alpha,\psi}u \in C([0, b], \mathbb{R})$.

Organization of the paper is that in Section 2, some preliminaries that are found to be useful in the present investigation are included. Section 3 and Section 4 are devoted to the study of existence, uniqueness of solution and estimates on solutions of the equations (3)–(4). In Section 5, we discuss the continuous dependence of solutions to equations (3)–(4). As an application of main results, two illustrative examples are given in last section.

2. Preliminaries

This section deals with relevant pre-requisties that are essential for investigations in the paper. We establish some significant results that will be needed to prove our main results.

Let $C[a, b]$ be the space of all continuous functions h from $[a, b]$ into $\mathbb{R}$ with the supremum (uniform) norm $\|h\|_{C([a,b],\mathbb{R})} = \sup\{|h(t)| : t \in [a, b]\}$. As usual, $C^n[a, b]$ denotes the space of n-times continuously differentiable functions on $[a, b]$. By $L^p[a, b]$ $p \geq 1$ we denote the space of all measurable functions $h : [a, b] \to \mathbb{R}$ with the norm $\|h\|_{L^p[a,b]} = \left(\int_a^b |h(t)|^p\, dt\right)^{\frac{1}{p}}$.

Definition 2.1. *([7], Sec.2.5, p.99) Let $\alpha > 0$ and ψ be an increasing function, having a continuous derivative ψ' on (a, b). Then the left and right-sided fractional integrals for an integrable function $h : [a, b] \longrightarrow \mathbb{R}$ with respect to ψ are defined by*

$$(I_{a^+}^{\alpha,\psi}h)(t) := \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1}h(s)ds,$$

$$(I_{b^-}^{\alpha,\psi}h)(t) := \frac{1}{\Gamma(\alpha)} \int_t^b \psi'(s)(\psi(s) - \psi(t))^{\alpha-1}h(s)ds.$$

Note that when $\psi(t) = t$, we obtain the known classical Riemann-Liouville fractional integral.

Definition 2.2. *([7]) Let $\alpha > 0$ is a real, $h : [a, b] \longrightarrow \mathbb{R}$ an integrable function and $\psi \in C^n[a, b]$ an increasing function such that $\psi'(t) \neq 0$, for all $t \in [a, b]$. Then the left-sided $\psi-$Riemann-Liouville fractional derivative of* *h of order α is given by*

$$\begin{aligned}(D_{a^+}^{\alpha,\psi}h)(t) &= \left[\frac{1}{\psi'(t)}\frac{d}{dt}\right]^n I_{a^+}^{n-\alpha,\psi}h(t)\\ &= \frac{1}{\Gamma(n-\alpha)}\left[\frac{1}{\psi'(t)}\frac{d}{dt}\right]^n \int_0^t \psi'(s)(\psi(t)-\psi(s))^{n-\alpha-1}h(s)ds,\end{aligned}$$

where $n=[\alpha]+1$, and $[\alpha]$ denotes the integer part of the real number α.

Definition 2.3. *([3], Def.1) Let $\alpha>0$, n be the smallest integer greater than or equal to α (i.e. $n=[\alpha]+1$), $h\in C^{n-1}[a,b]$, and ψ be given as in Definition 2.2. Then the left-sided $\psi-$Caputo fractional derivative of h of order α is defined as follows:*

$$({}^cD_{a^+}^{\alpha,\psi}h)(t) = D_{a^+}^{\alpha,\psi}\left[h(t)-\sum_{k=0}^{n-1}\frac{h_\psi^{[k]}(a)}{k!}(\psi(t)-\psi(a))^k\right],$$

where $h_\psi^{[k]}(t) = \left[\frac{1}{\psi'(t)}\frac{d}{dt}\right]^k h(t)$. Further, if $\alpha=n\in\mathbb{N}$, then ${}^cD_{a^+}^{\alpha,\psi}h(t) = h_\psi^{[n]}(t)$. In particular, if $0<\alpha<1$, then

$$({}^cD_{a^+}^{\alpha,\psi}h)(t) = D_{a^+}^{\alpha,\psi}\left[h(t)-h(a)\right].$$

Lemma 2.4. *([2]) Let $\alpha>0$. Given a function $h\in C^{n-1}[a,b]$, we have*

$$I_{a^+}^{\alpha,\psi}\,{}^cD_{a^+}^{\alpha,\psi}h(t) = h(t)-\sum_{k=0}^{n-1}\frac{h_\psi^{[k]}(a)}{k!}(\psi(t)-\psi(a))^k.$$

In particular, if $0<\alpha<1$, we have $I_{a^+}^{\alpha,\psi}\,{}^cD_{a^+}^{\alpha,\psi}h(t) = h(t)-h(a)$.

Lemma 2.5. *([7]) Let $\alpha>0$, $h\in C[a,b]$, and $\psi\in C^1[a,b]$. Then for all $t\in[a,b]$,*

(1) $I_{a^+}^{\alpha;\psi}(\cdot)$ *maps $C[a,b]$ into $C[a,b]$.*
(2) $I_{a^+}^{\alpha;\psi}h(a) = \lim_{t\to a^+} I_{a^+}^{\alpha;\psi}h(t) = 0.$

Lemma 2.6. *Let $\alpha,\beta>0$. Then*

(1) $I_{a^+}^{\alpha,\psi}(\psi(t)-\psi(a))^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}\left[\psi(t)-\psi(a)\right]^{\alpha+\beta-1}.$
(2) ${}^cD_{a^+}^{\alpha,\psi}(\psi(t)-\psi(a))^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\left[\psi(t)-\psi(a)\right]^{\beta-\alpha-1}.$
(3) ${}^cD_{a^+}^{\alpha,\psi}(\psi(t)-\psi(a))^k = 0$, $\forall k\in\{0,1,...,n-1\}$, $n\in\mathbb{N}$.

Proof. See ([2],[7]). □

Lemma 2.7. *Let $\alpha>0$ and $\beta>0$. Then the relation*

$$\left(I_{a^+}^{\alpha,\psi}I_{a^+}^{\beta,\psi}h\right)(t) = \left(I_{a^+}^{\alpha+\beta,\psi}h\right)(t)$$

holds almost everywhere for $t\in[a,b]$, for $h\in L^p[a,b]$ and $p\geq1$. If $\alpha+\beta>1$, then the relation holds at any point of $[a,b]$.

Proof. By definition 2.1 and using Dirichlet's formula, we have

$$\begin{aligned}
I_{a^+}^{\alpha,\psi} I_{a^+}^{\beta,\psi} h(t) &= \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\, I_{a^+}^{\beta,\psi} h(s) ds \\
&= \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s) \left[\psi(t)-\psi(s)\right]^{\alpha-1} \\
&\quad \times \left[\frac{1}{\Gamma(\beta)} \int_a^s \psi'(\tau) \left[\psi(s)-\psi(\tau)\right]^{\beta-1} h(\tau) d\tau\right] ds \\
&= \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(\beta)} \int_a^t \psi'(s) \left[\psi(t)-\psi(s)\right]^{\alpha-1} \\
&\quad \times \left[\int_\tau^t \psi'(\tau) \left[\psi(s)-\psi(\tau)\right]^{\beta-1} h(\tau) ds\right] d\tau \\
&= \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(\beta)} \int_a^t \psi'(\tau) h(\tau) \\
&\quad \times \left[\int_\tau^t \left[\psi(t)-\psi(s)\right]^{\alpha-1} \left[\psi(s)-\psi(\tau)\right]^{\beta-1} \psi'(s) ds\right] d\tau.
\end{aligned}$$

The inner integral is evaluated by the change of variable $\psi(s) = \psi(\tau) + z\left[\psi(t)-\psi(\tau)\right]$, this implies $\psi'(s)ds = \left[\psi(t)-\psi(\tau)\right] dz$. So,

$$\begin{aligned}
&\int_\tau^t \left[\psi(t)-\psi(s)\right]^{\alpha-1} \left[\psi(s)-\psi(\tau)\right]^{\beta-1} \psi'(s) ds \\
&= \left[\psi(t)-\psi(\tau)\right]^{\alpha+\beta-1} \int_0^1 (1-z)^{\alpha-1} z^{\beta-1} dz \\
&= \left[\psi(t)-\psi(\tau)\right]^{\alpha+\beta-1} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&I_{a^+}^{\alpha,\psi} I_{a^+}^{\beta,\psi} h(t) \\
&= \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(\beta)} \int_a^t \psi'(\tau) h(\tau) \left[\left[\psi(t)-\psi(\tau)\right]^{\alpha+\beta-1} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\right] d\tau \\
&= \frac{1}{\Gamma(\alpha+\beta)} \int_a^t \psi'(\tau) \left[\psi(t)-\psi(\tau)\right]^{\alpha+\beta-1} h(\tau) d\tau \\
&= I_{a^+}^{\alpha+\beta,\psi} h(t).
\end{aligned}$$

□

Theorem 2.8. *([5], Th.1.1) (Banach fixed point theorem) Let (X, d) be a complete metric space and $T : X \to X$ be a contraction mapping. Then T has a unique fixed point in X.*

3. Existence of the equations and existence of interval

In this section, we establish the existence of a subinterval where the solution of the considered problem exists and unique. We first give the following important results through which we can prove our main results.

Lemma 3.1. *Let $\alpha > 0$, ψ be as given in Definition 2.2, and $u \in C[0,b]$. Then*

$$^{c}D_{0^+}^{\alpha,\psi} I_{0^+}^{\alpha,\psi} u(t) = u(t), \ a.e.$$

Proof. From the Definition 2.3, we observe that

$$^{c}D_{0^+}^{\alpha,\psi} I_{0^+}^{\alpha,\psi} u(t) = D_{0^+}^{\alpha,\psi} \left[I_{0^+}^{\alpha,\psi} u(t) - \sum_{k=0}^{n-1} \frac{(I_{0^+}^{\alpha,\psi} u)_{\psi}^{[k]}(0)}{k!} (\psi(t) - \psi(0))^k \right].$$

Also, we have

$$\begin{aligned} (I_{0^+}^{\alpha,\psi} u)_{\psi}^{[k]}(t) &= \left[\frac{1}{\psi'(t)} \frac{d}{dt} \right]^k I_{0^+}^{\alpha,\psi} u(t) \\ &= D_{0^+}^{k,\psi} I_{0^+}^{k,\psi} I_{0^+}^{\alpha-k,\psi} u(t) \\ &= \frac{1}{\Gamma(\alpha - k)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-k-1} u(s) ds, \end{aligned}$$

hence we conclude that

$$\left| (I_{0^+}^{\alpha,\psi} u)_{\psi}^{[k]}(t) \right| \leq \frac{\|u\|_C}{\Gamma(\alpha - k + 1)} (\psi(t) - \psi(0))^{\alpha-k}.$$

Consequently, $(I_{0^+}^{\alpha,\psi} u)_{\psi}^{[k]}(0) = 0$, for all $k = 0, 1, ..., n-1$.

Thus,

$$\begin{aligned} ^{c}D_{0^+}^{\alpha,\psi} I_{0^+}^{\alpha,\psi} u(t) &= D_{0^+}^{\alpha,\psi} I_{0^+}^{\alpha,\psi} u(t) \\ &= \left[\frac{1}{\psi'(t)} \frac{d}{dt} \right]^n I_{0^+}^{n-\alpha,\psi} I_{0^+}^{\alpha,\psi} u(t) \\ &= D_{0^+}^{n,\psi} I_{0^+}^{n,\psi} u(t) = u(t). \end{aligned}$$

□

Corollary 3.2. *Let $\alpha > 0$, ψ be as given in Definition 2.2, and $h \in C[0,b]$. Then the integral equation*

$$u(t) = u_0 + I_{0^+}^{\alpha,\psi} h(t), \ t \in [0,b]. \tag{5}$$

is equivalent to the initial value problem

$$\begin{cases} ^{c}D_{0^+}^{\alpha,\psi} u(t) = h(t), \ a.e. t \in [0,b] \\ u(0) = u_0. \end{cases} \tag{6}$$

Proof. In fact, from the equation (5), Lemma 2.6 and Lemma 3.1, it follows that

$$\begin{aligned} ^{c}D_{0^+}^{\alpha,\psi} u(t) &= {}^{c}D_{0^+}^{\alpha,\psi} I_{0^+}^{\alpha,\psi} h(t) \\ &= h(t), \ a.e. \ t \in [0,b]. \end{aligned}$$

On the other hand, operating the Riemann–Liouville fractional integra operator $I_{a^+}^{\alpha,\psi}$ on both sides of the equation (6) and using Lemma 2.4, we get

$$u(t) = u_0 + I_{0^+}^{\alpha,\psi} h(t), \ t \in [0,b].$$

□

Corollary 3.3. *Let* $f : [0,b] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$, $g : C([0,b],\mathbb{R}) \to \mathbb{R}$ *are continuous and* ψ *be as given in Definition 2.2. Then the function* u *is given by*

$$u(t) = u_0 - g(u) + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} \times f(s,u(s), {}^c D_{0+}^{\alpha,\psi} u(s))ds. \tag{7}$$

Next, before stating and proving the main results, we introduce the following hypotheses.

(H_1) $f : [0,b] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous function and there exist $L_1 \in (0,+\infty)$, $L_2 \in (0,1)$ such that,

$$|f(t,x,y) - f(t,x^*,y^*)| \le L_1 |x - x^*| + L_2 |y - y^*|,$$

for all $t \in [0,b]$ and $x, x^*, y, y^* \in \mathbb{R}$.

(H_2) $g : C([0,b],\mathbb{R}) \to \mathbb{R}$ is continuous and there exists $L_3 \in (0,1)$ such that

$$|g(x) - g(x^*)| \le L_3 |x - x^*|, \ \forall x, x^* \in C([0,b],\mathbb{R}).$$

(H_3) $\psi \in C^1([0,b],\mathbb{R})$ an increasing bijective function and there exists $0 < \eta < b$ such that

$$\eta < \min \left\{ b, \psi^{-1} \left[\psi(0) + \left(\frac{(1-L_3)(1-L_2)\Gamma(\alpha+1)}{L_1} \right)^{\frac{1}{\alpha}} \right] \right\}. \tag{8}$$

Theorem 3.4. *Let the hypotheses* $(H_1), (H_2)$ *and* (H_3) *be satisfied. Then the equations (3)–(4) have a unique solution on the subinterval* $[0,\eta]$.

Proof. Define the operator $T : C([0,\eta],\mathbb{R}) \to C([0,\eta],\mathbb{R})$ as follows

$$(Tu)(t) = u_0 - g(u) + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} \times f(s,u(s), {}^c D_{0+}^{\alpha,\psi} u(s))ds.$$

For all $t \in [0,\eta]$ and for any $u, u^* \in C([0,\eta],\mathbb{R})$, we have

$$\begin{aligned} |Tu(t) - Tu^*(t)| &\le |g(u) - g(u^*)| + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} \\ &\quad \times \left| f(s,u(s), {}^c D_{0+}^{\alpha,\psi} u(s)) - f(s,u^*(s), {}^c D_{0+}^{\alpha,\psi} u^*(s)) \right| ds \\ &\le L_3 |u - u^*| + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} \\ &\quad \times L_2 |u - u^*| + L_2 \left| {}^c D_{0+}^{\alpha,\psi} u(s) - {}^c D_{0+}^{\alpha,\psi} u^*(s) \right| ds. \end{aligned} \tag{9}$$

Note that,

$$\begin{aligned} &\left| {}^c D_{0+}^{\alpha,\psi} u(s) - {}^c D_{0+}^{\alpha,\psi} u^*(s) \right| \\ &= \left| f(s,u(s), {}^c D_{0+}^{\alpha,\psi} u(s)) - f(s,u^*(s), {}^c D_{0+}^{\alpha,\psi} u^*(s)) \right| \\ &\le L_1 |u - u^*| + L_2 \left| {}^c D_{0+}^{\alpha,\psi} u(s) - {}^c D_{0+}^{\alpha,\psi} u^*(s) \right| \\ &\le \frac{L_1}{1-L_2} |u - u^*|. \end{aligned} \tag{10}$$

By invoking equation (10) in equation (9), we get

$$\begin{aligned} |Tu(t) - Tu^*(t)| \;\; &\leq \;\; L_3 \left| u - u^* \right| + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \\ &\quad \times L_2 \left| u - u^* \right| + \frac{L_2 L_1}{1 - L_2} \left| u - u^* \right| ds. \end{aligned}$$

Therefore,

$$\begin{aligned} & \|Tu - Tu^*\| \\ \leq \;\; & L_3 \left\| u - u^* \right\| + \frac{L_1 \left\| u - u^* \right\|}{(1 - L_2)\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} ds \\ = \;\; & \left[L_3 + \frac{L_1(\psi(t) - \psi(0))^\alpha}{(1 - L_2)\Gamma(\alpha + 1)} \right] \left\| u - u^* \right\| \\ \leq \;\; & \left[L_3 + \frac{L_1(\psi(\eta) - \psi(0))^\alpha}{(1 - L_2)\Gamma(\alpha + 1)} \right] \left\| u - u^* \right\|. \end{aligned}$$

In view of (H_3), we have

$$\eta < \psi^{-1} \left[\psi(0) + \left(\frac{(1 - L_3)(1 - L_2)\Gamma(\alpha + 1)}{L_1} \right)^{\frac{1}{\alpha}} \right]. \tag{11}$$

Further, since $\psi \in C^1([0, b], \mathbb{R})$ is a bijective function, $\psi^{-1} : \mathbb{R} \to [0, b]$ exists and from the inequality (11), we have

$$(\psi(\eta) - \psi(0))^\alpha < \left(\frac{(1 - L_3)(1 - L_2)\Gamma(\alpha + 1)}{L_1} \right).$$

So,

$$\frac{L_1 \left(\psi(\eta) - \psi(0) \right)^\alpha}{(1 - L_2)\Gamma(\alpha + 1)} < 1 - L_3,$$

which yields that

$$L_3 + \frac{L_1 \left(\psi(\eta) - \psi(0) \right)^\alpha}{(1 - L_2)\Gamma(\alpha + 1)} < 1.$$

Therefore,

$$\|Tu - Tu^*\| \leq \sigma \left\| u - u^* \right\|, \; 0 < \sigma < 1,$$

where

$$\sigma = L_3 + \frac{L_1 \left(\psi(\eta) - \psi(0) \right)^\alpha}{(1 - L_2)\Gamma(\alpha + 1)}.$$

This shows that T is a contraction mapping in $C([0, \eta], \mathbb{R})$. As consequence of Theorem 2.8, there is a unique fixed point $u \in C([0, \eta], \mathbb{R})$ such that $Tu = u$, which means that the equations (3)-(4) have a unique solution on $[0, \eta] \subset [0, b]$, and the proof is completed. □

4. Estimates on the solutions

Theorem 4.1. *Assume that $f : [0, b] \times \mathbb{R} \times \mathbb{R}$ and $g : C([0, b], \mathbb{R}) \rightarrow \mathbb{R}$ satisfying (H_1) and (H_2), let ψ be given as in Definition 2.1. If $u(t), t \in [0, b]$ any solution of the equations (3)-(4), then*

$$\|u\| \leq \frac{1}{\left[1 - L_3 - \frac{L_1}{1-L_2}\frac{(\psi(b)-\psi(0))^\alpha}{\Gamma(\alpha+1)}\right]} \times \left(|u_0| + \|g(0)\| + \frac{p_2}{1-L_2}\frac{(\psi(b)-\psi(0))^\alpha}{\Gamma(\alpha+1)}\right). \tag{12}$$

Provided that $\left[L_3 + \frac{L_1}{1-L_2}\frac{[\psi(b)-\psi(0)]^\alpha}{\Gamma(\alpha+1)}\right] < 1$, *where* $p_2 = \sup\{|f(t, 0, 0)| : t \in [0, b]\}$.

Proof. In view of Corollary 3.3, we have

$$u(t) = u_0 - g(u) + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \times f(s, u(s), {}^c D_{0^+}^{\alpha,\psi} u(s))ds.$$

Thus,

$$|u(t)| \leq |u_0| + |g(u) - g(0)| + |g(0)| + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1}\left|f(s, u(s), {}^c D_{0^+}^{\alpha,\psi} u(s))\right| ds$$

On the other hand, from (H_1) it follows for any $t \in [0, b]$, that

$$\begin{aligned} &\left|f(s, u(s), {}^c D_{0^+}^{\alpha,\psi} u(s))\right| \\ \leq &\left|f(s, u(s), {}^c D_{0^+}^{\alpha,\psi} u(s)) - f(s, 0, 0)\right| + \sup_{s\in[0,b]} |f(s, 0, 0)| \\ \leq &L_1 |u(s)| + L_2 \left|{}^c D_{0^+}^{\alpha,\psi} u(s)\right| + p_2 \\ \leq &L_1 |u(s)| + L_2 \left|f(s, u(s), {}^c D_{0^+}^{\alpha,\psi} u(s))\right| + p_2 \end{aligned}$$

Hence,

$$\left|f(s, u(s), {}^c D_{0^+}^{\alpha,\psi} u(s))\right| \leq \frac{1}{1-L_2}\left[L_1 |u(s)| + p_2\right].$$

Therefore,

$$|u(t)| \leq |u_0| + L_3 |u(t)| + |g(0)| + \frac{1}{1-L_2}\left[L_1 |u(s)| + p_2\right]\frac{(\psi(b) - \psi(0))^\alpha}{\Gamma(\alpha+1)}.$$

It yields that

$$\|u\| \leq |u_0| + L_3 \|u\| + \|g(0)\| + \frac{L_1 \|u\|}{1-L_2}\frac{(\psi(b) - \psi(0))^\alpha}{\Gamma(\alpha+1)} + \frac{p_2}{1-L_2}\frac{(\psi(b) - \psi(0))^\alpha}{\Gamma(\alpha+1)}.$$

Thus

$$\left[1 - L_3 - \frac{L_1}{1-L_2}\frac{(\psi(b)-\psi(0))^\alpha}{\Gamma(\alpha+1)}\right]\|u\|$$
$$\leq |u_0| + \|g(0)\| + \frac{p_2}{1-L_2}\frac{(\psi(b)-\psi(0))^\alpha}{\Gamma(\alpha+1)}.$$

$$\Leftrightarrow \quad \|u\| \leq \frac{1}{\left[1 - L_3 - \frac{L_1}{1-L_2}\frac{(\psi(b)-\psi(0))^\alpha}{\Gamma(\alpha+1)}\right]}$$
$$\times\left(|u_0| + \|g(0)\| + \frac{p_2}{1-L_2}\frac{(\psi(b)-\psi(0))^\alpha}{\Gamma(\alpha+1)}\right).$$

The proof is complete. □

5. Continuous dependence

In this section, we investigate the influence of a perturbation of initial data to the solutions of implicit fractional differential equation involving Caputo derivative with respect to another function ψ.

Firstly, we discuss the continuous dependence of solutions to the equations (3)-(4) on initial values. For this purpose, let u be a solution to the equations (3)-(4) and u^* be a solution of the following equations

$$ {}^cD_{0^+}^{\alpha;\psi}u^*(t) = f(t, u^*(t), {}^cD_{0^+}^{\alpha;\psi}u^*(t)), \quad 0 < \alpha < 1, \tag{13}$$

$$u^*(0) + g(u^*) = u_0^*. \tag{14}$$

Then we have the following result:

Theorem 5.1. *Assume that hypotheses of Theorem 3.4 hold. Let u and u^* are solutions of (3)-(4) and (13)-(14) respectively. Then there exists a constant λ such that*

$$|u(t) - u^*(t)| \leq \lambda\,|u_0 - u_0^*| \ , t \in [0, \eta].$$

Proof. In view of Theorem 3.4, we have

$$u(t) = u_0 - g(u) + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}$$
$$\times f(s, u(s), {}^cD_{0^+}^{\alpha;\psi}u(s))ds$$

and

$$u^*(t) = u_0^* - g(u^*) + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}$$
$$\times f(s, u^*(s), {}^cD_{0^+}^{\alpha;\psi}u^*(s))ds.$$

Then, for any t with $0 < t \leq \eta < b$, we get

$$\begin{aligned}
&|u(t) - u^*(t)| \\
\leq\ & |u_0 - u_0^*| + |g(u) - g(u^*)| + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \\
&\times \left| f(s, u(s), {}^c D_{0^+}^{\alpha;\psi} u(s)) - f(s, u^*(s), {}^c D_{0^+}^{\alpha;\psi} u^*(s)) \right| ds \\
\leq\ & |u_0 - u_0^*| + L_3 |u - u^*| + \frac{(\psi(t) - \psi(0))^\alpha}{\Gamma(\alpha+1)} \\
&\times \left(L_1 |u - u^*| + L_2 \left| {}^c D_{0^+}^{\alpha;\psi} u(s) - {}^c D_{0^+}^{\alpha;\psi} u^*(s) \right| \right) \\
\leq\ & |u_0 - u_0^*| + L_3 |u - u^*| + \frac{(\psi(t) - \psi(0))^\alpha}{\Gamma(\alpha+1)} \left(\frac{L_1}{1 - L_2} \right) |u - u^*| \\
\leq\ & |u_0 - u_0^*| + \left(L_3 + \frac{L_1(\psi(\eta) - \psi(0))^\alpha}{(1 - L_2)\Gamma(\alpha+1)} \right) |u - u^*| .
\end{aligned}$$

Thus, by (H_2), and take $\lambda = \frac{1}{1 - \left(L_3 + \frac{L_1(\psi(\eta) - \psi(0))^\alpha}{(1-L_2)\Gamma(\alpha+1)} \right)}$, we get

$$|u(t) - u^*(t)| \leq \lambda |u_0 - u_0^*| .$$

This completes the proof. □

Next we look at the influence of changes in the given function on the right-hand side of the implicit fractional differential equation. We denote by u the solution to equations (3)-(4) and by u^* the solution to the equations

$$(15) \qquad {}^c D_{0^+}^{\alpha;\psi} u^*(t) = \widetilde{f}(t, u^*(t), {}^c D_{0^+}^{\alpha;\psi} u^*(t)), \quad 0 < \alpha < 1,$$

$$(16) \qquad u^*(0) + g(u^*) = u_0.$$

Theorem 5.2. *Let $f, \widetilde{f}$ and g fulfill hypotheses (H_1) and (H_2). If u be a solution of equations (3)-(4) and u^* be a solution of equations (15)-(16). Then for $0 < t \leq \eta < b$, there exists a constant ρ such that*

$$|u(t) - u^*(t)| \leq \rho \sup_{(t,x,y) \in [0,\eta] \times \mathbb{R} \times \mathbb{R}} \left| f(t, x, y) - \widetilde{f}(t, x, y) \right| .$$

Proof. The equations (3)-(4) and (15)-(16), have similar integral solutions that are given by

$$\begin{aligned}
u(t) &= u_0 - g(u) + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \\
&\quad \times f(s, u(s), {}^c D_{0^+}^{\alpha;\psi} u(s)) ds
\end{aligned}$$

and

$$\begin{aligned}
u^*(t) &= u_0 - g(u^*) + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \\
&\quad \times \widetilde{f}(s, u^*(s), {}^c D_{0^+}^{\alpha;\psi} u^*(s)) ds,
\end{aligned}$$

respectively. It follows that

$$\begin{aligned}
& |u(t)-u^*(t)| \\
\leq\ & |g(u)-g(u^*)| + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} \\
& \times \left| f(s,u(s),{}^cD_{0^+}^{\alpha;\psi}u(s)) - \widetilde{f}(s,u^*(s),{}^cD_{0^+}^{\alpha;\psi}u^*(s)) \right| ds \\
\leq\ & L_3\,|u-u^*| + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} \\
& \times \left| f(s,u(s),{}^cD_{0^+}^{\alpha;\psi}u(s)) - f(s,u^*(s),{}^cD_{0^+}^{\alpha;\psi}u^*(s)) \right| ds \\
& + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} \\
& \times \left| f(s,u^*(s),{}^cD_{0^+}^{\alpha;\psi}u^*(s)) - \widetilde{f}(s,u^*(s),{}^cD_{0^+}^{\alpha;\psi}u^*(s)) \right| ds \\
\leq\ & L_3\,|u-u^*| + \frac{L_1(\psi(t)-\psi(0))^{\alpha}}{(1-L_2)\Gamma(\alpha+1)}|u-u^*| \\
& + \frac{(\psi(t)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)} \sup_{(t,x,y)\in[0,\eta]\times\mathbb{R}\times\mathbb{R}} \left| f(t,x,y)-\widetilde{f}(t,x,y) \right| \\
\leq\ & \left(L_3 + \frac{L_1(\psi(\eta)-\psi(0))^{\alpha}}{(1-L_2)\Gamma(\alpha+1)} \right)|u-u^*| \\
& + \frac{(\psi(\eta)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)} \sup_{(t,x,y)\in[0,\eta]\times\mathbb{R}\times\mathbb{R}} \left| f(t,x,y)-\widetilde{f}(t,x,y) \right|.
\end{aligned}$$

Since $\left(L_3 + \frac{L_1(\psi(\eta)-\psi(0))^{\alpha}}{(1-L_2)\Gamma(\alpha+1)}\right) < 1$, then

$$\begin{aligned}
|u(t)-u^*(t)| \ \leq\ & \frac{(\psi(\eta)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)\left[1-\left(L_3 + \frac{L_1(\psi(\eta)-\psi(0))^{\alpha}}{(1-L_2)\Gamma(\alpha+1)}\right)\right]} \\
& \times \sup_{(t,x,y)\in[0,\eta]\times\mathbb{R}\times\mathbb{R}} \left| f(t,x,y)-\widetilde{f}(t,x,y) \right|.
\end{aligned}$$

Take

$$\rho = \frac{(\psi(\eta)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)\left[1-\left(L_3 + \frac{L_1(\psi(\eta)-\psi(0))^{\alpha}}{(1-L_2)\Gamma(\alpha+1)}\right)\right]},$$

we obtain

$$|u(t)-u^*(t)| \leq \rho \sup_{(t,x,y)\in[0,\eta]\times\mathbb{R}\times\mathbb{R}} \left| f(t,x,y)-\widetilde{f}(t,x,y) \right|.$$

The proof is completed. □

6. Examples

In this section we present some examples to illustrate our results.

Example 6.1. *Consider the implicit fractional differential equation*

$$(17) \qquad {}^cD_{0^+}^{\alpha,\psi}u(t) = \frac{1}{(1+8e^t)\left(1+|u(t)|+\left|{}^cD_{0^+}^{\alpha,\psi}u(t)\right|\right)},$$

with the nonlocal conditions

$$u(0)+\sum_{k=1}^{m}c_k u(t_k)=1, \tag{18}$$

where $0<t_1<t_2<...<t_m<1$, $c_k>0$ $(k=1,2,...,m)$ *with* $\sum_{k=1}^{m}c_k<\frac{2}{3}$, $\alpha=\frac{1}{2}$, $f(t,u(t),{}^cD_{0^+}^{\alpha,\psi}u(t))=\frac{1}{(1+8e^t)\left(1+|u(t)|+\left|{}^cD_{0^+}^{\alpha,\psi}u(t)\right|\right)}$, *and* $g(u)=\sum_{k=1}^{m}c_k u(t_k)$. *Let* $\mathbb{R}=\mathbb{R}^+$. *Then for* $u,u^*,v,v^*\in\mathbb{R}^+$ *and for* $t\in[0,1]$, *we can see that*

$$\begin{aligned} & |f(t,u,v)-f(t,u^*,v^*)| \\ = & \left|\frac{1}{(1+8e^t)(1+|u|+|v|)}-\frac{1}{(1+8e^t)(1+|u^*|+|v^*|)}\right| \\ \leq & \frac{1}{(1+8e^t)}\left|\frac{|u^*|+|v^*|-|u|-|v|}{(1+|u|+|v|)(1+|u^*|+|v^*|)}\right| \\ \leq & \frac{1}{9}\left[|u-u^*|+|v-v^*|\right] \end{aligned}$$

and

$$\begin{aligned} |g(u)-g(v)| &= \left|\sum_{k=1}^{m}c_k u(t_k)-\sum_{k=1}^{m}c_k v(t_k)\right| \\ &\leq \sum_{k=1}^{m}c_k\,|u-v|\,, \quad \forall u,v\in C([0,1],\mathbb{R}^+). \end{aligned}$$

Hence the conditions (H_1),(H_2) *hold with* $L_1=L_2=\frac{1}{9}$ *and* $L_3=\sum_{k=1}^{m}c_k<\frac{2}{3}$. *We shall check that condition* $\left[L_3+\frac{L_1}{1-L_2}\frac{(\psi(\eta)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)}\right]<1$. *Indeed, we take* $\psi(t):=2^t$ *for all* $t\in[0,1]$ *and choose* $\eta=\frac{1}{2}$. *Thus, a simple computation shows that*

$$\begin{aligned} & \sum_{k=1}^{m}c_k+\frac{\frac{1}{9}}{1-\frac{1}{9}}\frac{\left[\psi(\frac{1}{2})-\psi(0)\right]^{\frac{1}{2}}}{\Gamma(\frac{1}{2}+1)} \\ < & \frac{2}{3}+\frac{\frac{1}{9}}{1-\frac{1}{9}}\frac{\left[(2)^{\frac{1}{2}}-1\right]^{\frac{1}{2}}}{\Gamma(\frac{3}{2})}\approx 0.757<1. \end{aligned}$$

Further, the condition (H_3) *holds too, i.e.* $\psi(\frac{1}{2})-\psi(0)\approx 0.414<\min\{1,5.585\}$. *As consequence of Theorem 3.4, then the equations (17)-(18) have a unique solution on* $[0,\frac{1}{2}]\subset[0,1]$.

Finally, $\|g(0)\|=0$ *and* $p_2=\sup\{\left|\frac{1}{(1+8e^t)}\right|:t\in[0,1]\}=\frac{1}{9}$. *So by applying Theorem 4.1, we obtain*

$$\|u\|\leq\frac{1}{\left[1-\frac{2}{3}-\frac{\frac{1}{9}}{1-\frac{1}{9}}\frac{(2-1)^{\frac{1}{2}}}{\Gamma(\frac{1}{2}+1)}\right]}\left(1+0+\frac{\frac{1}{9}}{1-\frac{1}{9}}\frac{(2-1)^{\frac{1}{2}}}{\Gamma(\frac{1}{2}+1)}\right)\approx 5.934.$$

Example 6.2. *Consider an implicit nonlocal problem of fractional order*

$${}^cD_{0^+}^{\alpha,\psi}u(t)=t+\frac{1}{6}\cos u(t)+\frac{1}{2}\sin({}^cD_{0^+}^{\alpha,\psi}u(t)), \tag{19}$$

$$u(0)=\frac{1}{4}u(\frac{1}{3}), \tag{20}$$

where $u_0=0$, $\alpha=\frac{1}{3}$, $f(t,u(t),{}^cD^{\alpha,\psi}_{0^+}u(t))=t+\frac{1}{6}\cos u(t)+\frac{1}{2}\sin({}^cD^{\alpha,\psi}_{0^+}u(t))$, *and* $g(u)=\frac{1}{4}u(\frac{1}{3})$. *Let* $\mathbb{R}=\mathbb{R}^+$. *Then for* $u,u^*,v,v^*\in\mathbb{R}^+$ *and for* $t\in[0,1]$, *we find that*

$$|f(t,u,v)-f(t,u^*,v^*)|\leq\frac{1}{6}|u-u^*|+\frac{1}{2}|v-v^*|$$

and

$$|g(u)-g(v)|\leq\frac{1}{4}|u-v|,\quad\forall u,v\in C([0,1],\mathbb{R}^+).$$

Hence the conditions (H_1) *and* (H_2) *hold with* $L_1=\frac{1}{6}$, $L_2=\frac{1}{2}$ *and* $L_3=\frac{1}{4}$. *Now, we will check that* $\left[L_3+\frac{L_1}{1-L_2}\frac{(\psi(\eta)-\psi(0))^\alpha}{\Gamma(\alpha+1)}\right]<1$. *Set* $\psi(t):=\sqrt{1+t}$ *for all* $t\in[0,1]$, *we can choose* $\eta=\frac{1}{3}$. *So, some elementary computations gives us*

$$\left[L_3+\frac{L_1}{1-L_2}\frac{(\psi(\eta)-\psi(0))^\alpha}{\Gamma(\alpha+1)}\right]\approx 0.450<1.$$

Moreover, $\psi(\frac{1}{3})-\psi(0)\approx 0.155<\min\{1,8.111\}$ *which means the condition* (H_3) *holds. Thus all the conditions of Theorem 3.4 are satisfied. Hence, by the conclusion of Theorem 3.4, the equations (19)-(20) have a unique solution on* $[0,\frac{1}{3}]\subset[0,1]$. *Furthermore,* $\|g(0)\|=0$ *and* $p_2=\sup\{\left|t+\frac{1}{6}\right|:t\in[0,1]\}=\frac{7}{6}$. *Thus, by Theorem 4.1, we obtain*

$$\|u\|\leq\frac{1}{\left[1-\frac{1}{4}-\frac{\frac{1}{6}}{1-\frac{1}{2}}\frac{(\sqrt{2}-1)^{\frac{1}{3}}}{\Gamma(\frac{1}{3}+1)}\right]}\left(\frac{\frac{7}{6}}{1-\frac{1}{2}}\frac{(\sqrt{2}-1)^{\frac{1}{3}}}{\Gamma(\frac{1}{3}+1)}\right)\approx 4.129.$$

Conclusions

We can conclude that the main results of this article have been successfully achieved, that is, through Banach contraction mapping principle, extremely important results within the mathematical analysis, we scrutinized the existence, uniqueness, and estimates on solutions of the nonlocal Cauchy problem for nonlinear implicit fractional differential equation introduced by the ψ-Caputo fractional derivative. On the other hand, as application, the continuous dependence of the Cauchy-type problem on data was discussed. This paper contributes to the growth of the fractional calculus, especially in the case differential equations of fractional order involving a general formulation of Caputo fractional derivative with respect to function ψ. There are some articles that carried out a brief study on existence, uniqueness, and continuous dependence of solutions of nonlinear implicit differential equations with fractional order, and one of the objectives of this paper is to contribute so that it can have a greater extent of studies within the mathematical analysis related to implicit fractional differential equations.

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